LEC 11. Bessel Functions and vibrating circular membranes.

Vibrating circular membranes

$$u_{tt} = c^2 \Delta u$$
$$= c^2 \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right).$$

$\Omega = \{ (r,\theta) : 0 < r < a,\ -\pi < \theta \leq \pi \}$.

BC: $u(a,\theta,t) = 0$.
$|u(0,\theta,t)| < \infty$.

Separation of variables.

$$u(r,\theta,t) = \phi(r,\theta)\, h(t)$$

$$\frac{h''}{c^2 h} = \frac{1}{\phi}\left[\frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta} \right] = -\lambda$$

time equation $h'' + c^2 h = 0 \Rightarrow h = c_1 \cos\sqrt{\lambda}ct + c_2 \sin\sqrt{\lambda}ct$.

Space equation. $\begin{cases} \frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta} + \lambda\phi = 0. \\ \phi(a,\theta) = 0 \quad |\phi(0,\theta)| < \infty. \end{cases}$

Further separation. $\phi(r,\theta) = f(r) g(\theta)$.

$$\frac{1}{r} g(\theta)(r f'(r))' + \frac{1}{r^2} g''(\theta) f(r) + \lambda f g = 0$$

divide by $\frac{1}{r^2} fg$

$$\Rightarrow \frac{r(rf')'}{f} + \frac{g''}{g} + r^2\lambda = 0.$$

$$\Rightarrow \frac{r(rf')'}{f} + r^2\lambda = -\frac{g''}{g} = \mu.$$

$$g'' + \mu g = 0 \quad w/ \quad g(-\pi) = g(\pi) \quad g'(-\pi) = g'(\pi)$$

Eigenvalue problem in θ $\Rightarrow$ $g(\theta) = a\cos\sqrt{\mu}\,\theta + b\sin\sqrt{\mu}\,\theta$

$$\mu_m = m^2 \quad \text{for} \quad m = 0, 1, 2, \cdots$$

$$r(rf')' + (r^2\lambda - \mu)f = 0. \qquad f(a) = 0 \quad f(0) < \infty.$$

Eigenvalue problem in r.

Let $z = \sqrt{\lambda}\, r$

$$\frac{d}{dr} = \frac{d}{dz}\cdot\frac{dz}{dr} = \sqrt{\lambda}\cdot\frac{d}{dz}$$

$$\Rightarrow \boxed{z^2 f''(z) + 2f'(z) + (z^2 - m^2)f = 0.}$$

Bessel's equation of order 0.

$f(\sqrt{\lambda}\,a) = 0$ & $f(0) < \infty$.

By Rayleigh quotient, $\lambda > 0$. $0 < r < a$

$\Rightarrow 0 < z < \sqrt{\lambda}\, a$.

Note: We can write this equation in Sturm-Liouville form.

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)f = 0$$

$x = r \qquad p(r) = r \qquad \sigma(r) = r \qquad q(r) = -\frac{m^2}{r}$.

This is not a regular S-L problem.

1. BC at $r = 0$ is not regular.
2. $p(0) = 0$, $\sigma(0) = 0$ violate the condition $p, \sigma > 0$.

3. $q(r) \to -\infty$ as $r \to 0$ violates $q(r)$ has to be continuous at $r=0$.

It turns out the S-T thm is still true for this problem.

1. $\forall m$, $\exists$ infinitely many eigenvalues λ_{nm}, $n=0,1,2\ldots$ $n=1,2,\cdots$
2. $f_{nm}(r)$ is eigenfunction to the eigenvalue λ_{nm}.
3. Eigenfunctions are orthogonal w.r.t. weight r.

$$\int_0^a f_{n_1,m}(r)\, f_{n_2,m}(r)\, r\, dr = 0 \quad n_1 \neq n_2$$

Bessel's equation

$$z^2 f'' + zf' + (z^2 - m^2) f = 0.$$

$$f'' + \frac{1}{z} f' + \left(1 - \frac{m^2}{z^2}\right) f = 0. \quad \Rightarrow \ z=0 \text{ is a singular point.}$$

Behavior ~~of~~ near z_0. $\quad z^2 f \approx 0$

Note: zf' and $z^2 f''$ might not be small near $z=0$ since derivatives of f might be large near $z=0$.

$\therefore$ Around $z=0$, $\quad z^2 f'' + zf' - m^2 f \approx 0.$

$\Rightarrow$ equidimensional equation.

if $m>0$ $\quad f \approx C_1 z^m + C_2 z^{-m}$

if $m=0$ $\quad f \approx C_1 + C_2 \ln z.$

Bessel function

The exact solution to the Bessel equation can be written as $J_m(z)$ (Bessel function of the 1st kind of order m)

and Y_m (Bessel function of the 2nd kind of order m).

$$\Rightarrow f(z) = C_1 J_m(z) + C_2 Y_m(z).$$

Note:

1. $J_m(z)$ is well-behaved near $z=0$.

$$J_m(z) \sim \begin{cases} 1 & m=0 \\ \frac{1}{2^m m!} z^m, & m>0 \end{cases} \quad \text{as } z \to 0.$$

2. $Y_m(z)$ blows up at $z=0$.

$$Y_m(z) \sim \begin{cases} \frac{2}{\pi} \ln z & m=0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m} & m>0 \end{cases} \quad \text{as } z \to 0$$

Note: one cannot ~~with~~ write Bessel functions in terms of elementary functions (exp, cos, sin, etc.). But we can write ~~it in as~~ them as infinite series.
Infact,

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+m}}{k!\,(k+m)!}$$

Now, since $|f(0)| < \infty \Rightarrow C_2 = 0$.

$f(z) = C_1 J_m(z)$. $\quad f(\sqrt{\lambda}\, a) = 0 \Rightarrow J_m(\sqrt{\lambda}\, a) = 0$.
eigenvalues are related to the roots of the Bessel functions of the 1st kind.

Let z_{mn} be the nth root of $J_m(z)$.

$$\Rightarrow \sqrt{\lambda}\, a = z_{mn} \Rightarrow (\lambda_{mn}) = \left(\frac{z_{mn}}{a}\right)^2 :$$

For each $m = 0, 1, 2$, there are infinitely many eigenvalues $n = 1, 2, \cdots\cdots$

Eigenfunctions $f(z) = J_m\left(\frac{z_{mn}}{a} r\right)$ $m=0,1,\dots$

$n = 1,2,\dots$

<u>Orthogonality</u> $\int_0^a J_m\left(\frac{z_{mp}}{a} r\right) J_m\left(\frac{z_{mq}}{a} r\right) r\,dr = 0$

if $p \neq q$.

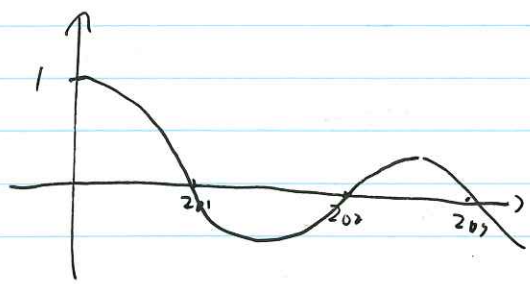
<u>Completeness</u> Any piecewise smooth function can be written as.

$$u(r) \sim \sum_{n=1}^{\infty} a_n J_m\left(\frac{z_{mn}}{a} r\right)$$: Fourier-Bessel series.

by orthogonality.

$$a_n = \frac{\int_0^a u(r) J_m\left(\frac{z_{mn}}{a} r\right) r\,dr}{\int_0^a J_m^2\left(\frac{z_{mn}}{a} r\right) r\,dr}$$

Ex. $J_0(z)$



$z_{01} \approx 2.4$

$z_{02} \approx 5.52$

$z_{03} \approx 8.65$

product solution.

$$J_m(\sqrt{\lambda_{mn}}\, r)\begin{Bmatrix}\cos m\theta \\ \sin m\theta\end{Bmatrix}\begin{Bmatrix}\cos c\sqrt{\lambda_{mn}}\, t \\ \sin c\sqrt{\lambda_{mn}}\, t\end{Bmatrix}$$

Back to circular membrane.

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J\left(\frac{z_{mn}}{a} r\right)\cos(m\theta)\cos\left(c\,\frac{z_{mn}}{a} t\right)$$

$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn} J\left(\frac{z_{mn}}{a} r\right)\sin(m\theta)\sin\left(c\,\frac{z_{mn}}{a} t\right).$$

IC: $u(r,\theta,0) = \alpha(r,\theta)$ $u_t(r,\theta,0) = 0$.

$$A_{0n} = \frac{\int_{-\pi}^{\pi} \int_0^a \alpha(r,\theta)\, J_0\left(\frac{z_{0n}}{a} r\right) r\, dr\, d\theta}{2\pi \int_0^a J_0^2\left(\frac{z_{0n}}{a} r\right) r\, dr}$$

$$A_{mn} = \frac{\int_{-\pi}^{-\pi} \int_0^a \alpha(r,\theta)\, J_m\left(\frac{z_{mn}}{a} r\right) \cos m\theta \; r\, dr\, d\theta}{\pi \int_0^a J_m^2\left(\frac{z_{mn}}{a} r\right) r\, dr}$$

$$B_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^a \alpha(r,\theta)\, J_m\left(\frac{z_{mn}}{a} r\right) \sin m\theta \; r\, dr\, d\theta}{\pi \int_0^a J_m^2\left(\frac{z_{mn}}{a} r\right) r\, dr}$$

<u>Circularly symmetric case</u>

Q: What if the IC is independent of θ?

Intuitively, the solution should be independent of θ.

$$\begin{cases} u_{tt} = c^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) & u = u(r,t) \\ u(a,t) = 0 & |u(0,t)| < \infty \\ u(r,0) = \alpha(r) & u_t(r,0) = \beta(r). \end{cases}$$

Separation of variables. $u(r,t) = \phi(r)h(t)$

$$\frac{h''}{c^2 h} = \frac{1}{r\phi}(r\phi')' = -\lambda$$

space problem: $\frac{d}{dr}\left(r \frac{d\phi}{dr}\right) + \lambda r \phi = 0.$

$$\phi(a) = 0 \quad |\phi(0)| < \infty.$$

$$\Rightarrow \quad r\phi'' + \phi' + \lambda r \phi = 0.$$

Let $z = \sqrt{\lambda}\, r$

$$\Rightarrow \quad z^2 \frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + z^2\phi = 0 \quad \Rightarrow \text{ Bessel eqn. of order } 0.$$

$$\therefore \quad \phi(z) = c_1 J_0(z) + c_2 Y_0(z)$$

$$|\phi(0)| < \infty \Rightarrow c_2 = 0.$$

$$\underset{\substack{\neq\\0}}{c_1} J_0(\sqrt{\lambda}a) = 0 \Rightarrow \sqrt{\lambda}a = z_{0n} \ : \ n\text{th root of } J_0(z).$$

$$\Rightarrow \lambda = \left(\frac{z_{0n}}{a}\right)^2. \qquad n = 1, 2, \cdots$$

$$\phi_n(r) = J_0\left(\frac{z_{0n}}{a} r\right).$$

Now

$$u(r,t) = \sum_{n=1}^{\infty} a_n J_0\left(\frac{z_{0n}}{a} r\right)\cos\left(\frac{z_{0n}}{a} ct\right) + b_n J_0\left(\frac{z_{0n}}{a} r\right)\sin\left(\frac{z_{0n}}{a} ct\right)$$

Exercise. Find a_n, b_n.

Note: The circularly symmetric solution is the $m=0$ part of the more general problem.